ON A THEOREM OF LANDAU AND TOEPLITZ

ROBERT B. BURCKEL, DONALD E. MARSHALL, AND PIETRO POGGI-CORRADINI

Abstract. The now canonical proof of Schwarz's Lemma appeared in a 1907 paper of Carathéodory, who attributed it to Erhard Schmidt. Since then, Schwarz's Lemma has acquired considerable fame, with multiple extensions and generalizations. Much less known is that, in the same year 1907, Landau and Toeplitz obtained a similar result where the diameter of the image set takes over the role of the maximum modulus of the function. We give a streamlined proof of this result and also extend it to include bounds on the growth of the maximum modulus.

1. Schwarz's Lemma

First, let us set the following standard notations: $\mathbb{C}$ denotes the complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Moreover, for $r > 0$, we let $r\mathbb{D} := \{z \in \mathbb{C} : |z| < r\}$ and $r\mathbb{T} := \{z \in \mathbb{C} : |z| = r\}$.

We begin by recalling the aforementioned

Theorem 1.1 (Schwarz's Lemma). *Suppose f is analytic on the unit disk $\mathbb{D}$ and*

$$\sup_{|z|<1} |f(z) - f(0)| \leq 1. \tag{1.1}$$

Then,

$$|f'(0)| \leq 1 \tag{1.2}$$

and

$$|f(z) - f(0)| \leq |z| \tag{1.3}$$

for every $z \in \mathbb{D}$.

Moreover, equality holds in (1.2) *or in* (1.3) *at some point in $\mathbb{D} \setminus \{0\}$ if and only if $f(z) = a + cz$ for some constants $a, c \in \mathbb{C}$ where $|c| = 1$.*

The standard way to prove Schwarz's Lemma is to factor $f(z) - f(0) = zg(z)$, for some analytic function g, then apply the maximum modulus theorem to g to deduce that $\sup_{|z|<1} |g(z)| \leq 1$. This argument first appeared in a paper of Carathèodory [C1907] where the idea is attributed to E. Schmidt. See Remmert, [R1991] p. 272-273, and Lichtenstein, [Li1919] footnote 427, for historical accounts.

Date: March 20, 2006.

1991 *Mathematics Subject Classification.* 30C80.

2. The theorem of Landau and Toeplitz

In a 1907 paper, Landau and Toeplitz prove a similar result where (1.1) is replaced by the diameter of the image set. (For a set $E \subset \mathbb{C}$ the diameter is $\operatorname{Diam} E := \sup_{z,w\in E} |z-w|$.)

Theorem 2.1 (Landau-Toeplitz [LaT1907]). *Suppose f is analytic on the unit disk $\mathbb{D}$ and $\operatorname{Diam} f(\mathbb{D}) \leq 2$. Then*

$$|f'(0)| \leq 1. \tag{2.1}$$

Moreover, equality holds in (2.1) if and only if $f(z) = a + cz$ for some $a, c \in \mathbb{C}$ where $|c| = 1$.

Remark 2.2. As we will see in the proof below the inequality (2.1) is a simple consequence of (1.2). The main hurdle is proving the case of equality. Inequality (2.1) appears in the classic book of Pólya and Szegö, p. 151 and p. 356 [PolS1972], where the paper of Landau and Toeplitz is mentioned. However, the case of equality is not discussed.

Remark 2.3. Notice, for instance, that Theorem 2.1 covers the case when $f(\mathbb{D})$ is an equilateral triangle of side-length 2, which is of course not contained in a disk of radius 1.

Proof. Decompose f into its odd part and its even part: $f(z) = f_o(z) + f_e(z)$, where $f_o(z) := (f(z) - f(-z))/2$ and $f_e(z) := (f(z) + f(-z))/2$. Then $|f_o(z)| \leq \operatorname{Diam} f(\mathbb{D})/2 \leq 1$, $f_o(0) = 0$ and $f_o'(0) = f'(0)$, so by Schwarz's Lemma (Theorem 1.1) applied to f_o we get

$$|f'(0)| \leq 1.$$

Theorem 2.1 is a consequence of the following claim.

Claim 2.4. *If $|f'(0)| = 1$, then $f(z) \equiv f(0) + f'(0)z$.*

From the 'equality' part of Schwarz's Lemma (Theorem 1.1), we find that

$$f_o(z) = \frac{f(z) - f(-z)}{2} = f'(0)z \qquad \forall z \in \mathbb{D} \tag{2.2}$$

We use (2.2) to show that f is linear.

For $0 \leq r < 1$, let $D_r := \operatorname{Diam} f(r\mathbb{D})$. First, we show that $D_r = \operatorname{Diam} f(r\mathbb{T})$. Indeed, notice that by the Open Mapping Theorem the set

$$\{f(z) - f(w) : |z| < r,\ |w| < r\} = \bigcup_{|w|<r} [f(r\mathbb{D}) - f(w)]$$

is open. Therefore, no number in this set has modulus D_r. However, there are points $z_0, w_0 \in r\overline{\mathbb{D}}$ with $|f(z_0) - f(w_0)| = D_r$. So one at least of them, say w_0, must lie on $r\mathbb{T}$. But then since $f(r\mathbb{D}) - f(w_0)$ is open, z_0 cannot belong to $r\mathbb{D}$. Thus both z_0, w_0 lie on $r\mathbb{T}$, which proves that $D_r = \operatorname{Diam} f(r\mathbb{T})$.

Now we show that the diameter of the image grows linearly, more precisely, $D_r = 2r$ for every $0 \le r < 1$.

Since

$$h_u(z) := \frac{f(z) - f(-uz)}{z}$$

is an analytic function for $z \in \mathbb{D}$ whenever $u \in \mathbb{T}$ is fixed, its maximum modulus on the disk $r\overline{\mathbb{D}}$ is either constant in r or increasing in r. So the quantity

$$\frac{D_r}{r} = \frac{\operatorname{Diam} f(r\mathbb{T})}{r} = \max_{|z|=r} \max_{|u|=1} \left| \frac{f(z) - f(-uz)}{z} \right| = \max_{|u|=1} \max_{|z|\le r} |h_u(z)|$$

is also either constant in r or increasing in r. But, on one hand,

$$\frac{D_r}{r} \ge \sup_{|u|=1} |h_u(0)| = \sup_{|u|=1} |1+u||f'(0)| = 2.$$

And, on the other hand, since $D_r/r \le D_1/r \le 2/r$,

$$\lim_{r\uparrow 1} \frac{D_r}{r} \le 2.$$

So D_r/r must be constant and $D_r = 2r$ for every $0 < r < 1$.

Now, for every $|w| < 1$, consider the function

$$g_w(z) := \frac{f(z) - f(-w)}{2f'(0)}$$

which is analytic for $z \in \mathbb{D}$. Then, by (2.2), $g_w(w) = w$. Also, if $0 < |w| = r < 1$,

$$|g_w(w)| = r = \frac{D_r}{2} \ge \sup_{|z|<r} |g_w(z)|$$

i.e., g_w fixes w and preserves the disk $D(0, |w|)$ centered at 0 and of radius $|w|$. Using Lemma 2.5 below, we get that $\operatorname{Im} g_w'(w) = 0$. Therefore,

$$\operatorname{Im} \frac{f'(w)}{2f'(0)} = 0 \qquad \forall w \in \mathbb{D}, \tag{2.3}$$

whence, thanks to the Open Mapping Theorem, $f'(w)/(2f'(0))$ is constant and equal to $1/2$. Thus,

$$f(z) \equiv f(0) + f'(0)z \qquad \forall z \in \mathbb{D},$$

which proves Claim 2.4 and hence Theorem 2.1. □

We are left to show the following lemma.

Lemma 2.5. *Suppose g is analytic in $\mathbb{D}$, $0 < r < 1$, $|w| = r$ and*

$$w = g(w) \text{ and } r = \max_{|z|=r} |g(z)|.$$

Then, $\operatorname{Im} g'(w) = 0$.

Proof. Actually, the stronger conclusion $g'(w) \geq 0$ is geometrically obvious because when $g'(w) \neq 0$, the map g is very close to the rotation-dilation centered at w given by $\zeta \mapsto w + g'(w)(\zeta - w)$. But since g can't rotate points inside $D(0, |w|)$ to a point outside, the derivative must be positive.

For the sake of rigor, we instead give a "calculus" proof of the weaker statement, which has the advantage of being more historically accurate, since it can be perceived in the original paper of Landau and Toeplitz, and which they credit to F. Hartogs.

For $\theta \in \mathbb{R}$ introduce

$$\phi(\theta) := |g(we^{i\theta})|^2 = g(we^{i\theta})\overline{g(we^{i\theta})}.$$

The function $g^\star(z) := \overline{g(\bar{z})}$ is also analytic in $\mathbb{D}$, and ϕ may be written

$$\phi(\theta) = g(we^{i\theta})g^\star(\bar{w}e^{-i\theta}),$$

enabling us to compute $\phi'(\theta)$ via the product and chain rules. We get routinely,

$$\phi'(\theta) = -2\,\mathrm{Im}\left[we^{i\theta}g'(we^{i\theta})\overline{g(we^{i\theta})}\right]$$

and setting $\theta = 0$,

$$\phi'(0) = -2\,\mathrm{Im}\left[wg'(w)\overline{g(w)}\right] = -2\,\mathrm{Im}\left[wg'(w)\overline{w}\right] = -2|w|^2\,\mathrm{Im}\,g'(w).$$

Since ϕ realizes its maximum over $\mathbb{R}$ at $\theta = 0$, we have $\phi'(0) = 0$, so the preceding equality proves Lemma 2.5. $\square$

3. Some corollaries of the Landau-Toeplitz Theorem

Corollary 3.1. *Suppose f is analytic on $\mathbb{D}$. If $D_r = \mathrm{Diam}\, f(r\mathbb{D})$ is linear in r, then f is linear.*

Proof. By continuity, $D_r = cr$ for some $c > 0$. Then $g := 2f/c$ satisfies $\mathrm{Diam}\, g(\mathbb{D}) = 2$ and by rotating and translating we get $0 < g'(0) \leq 1$, as in the initial invocation of Schwarz's Lemma in the proof of the Landau-Toeplitz Theorem 2.1. However, for r small $g(r\mathbb{D})$ is almost round, hence

$$2r = D_r = \mathrm{Diam}\, g(r\mathbb{D}) = 2rg'(0) + o(r)$$

i.e., $g'(0) = 1$. Now apply Claim 2.4. $\square$

Recall that for a simply-connected planar domain Ω (not $\mathbb{C}$) the hyperbolic density ρ_Ω is defined on Ω so that

$$\rho_\Omega(w)|dw| = \rho_\Omega(f(z))|f'(z)||dz| = \rho_{\mathbb{D}}(z)|dz| := \frac{|dz|}{1 - |z|^2}$$

for some, and hence for every, conformal map f of $\mathbb{D}$ onto Ω. By Schwarz's Lemma (Theorem 1.1), the following monotonicity holds

$$\Omega \subset \tilde{\Omega} \qquad \Longrightarrow \qquad \rho_\Omega(z) \geq \rho_{\tilde{\Omega}}(z) \qquad \forall z \in \Omega.$$

Also, one can check that if Ω is bounded, then ρ_Ω always attains its minimum.

Corollary 3.2. *Every simply-connected planar domain* Ω *with* $\operatorname{Diam}\Omega \le 2$ *satisfies*

$$\min_{w\in\Omega} \rho_\Omega(w) \ge 1. \tag{3.1}$$

Moreover, equality holds in (3.1) *if and only if* Ω *is a disk of radius* 2.

Proof. Fix $w \in \Omega$. By the Riemann mapping theorem there is a one-to-one and analytic map f of $\mathbb{D}$ onto Ω such that $f(0) = w$. Then $\rho_\Omega(w) = 1/|f'(0)|$. Now apply the Landau-Toeplitz Theorem 2.1. □

4. Growth bounds

In view of the growth bound (1.3) in Schwarz's Lemma, it is natural to ask whether a similar statement holds in the context of 'diameter'. We offer the following result.

Theorem 4.1. *Suppose* f *is analytic on the unit disk* $\mathbb{D}$ *and* $\operatorname{Diam} f(\mathbb{D}) \le 2$. *Then for all* $z \in \mathbb{D}$

$$|f(z) - f(0)| \le |z| \frac{2}{1 + \sqrt{1 - |z|^2}}. \tag{4.1}$$

Moreover, equality holds in (4.1) *at some point in* $\mathbb{D}\setminus\{0\}$ *if and only if* f *is a linear fractional transformation of the form*

$$f(z) = c\frac{z - b}{1 - \bar{b}z} + a \tag{4.2}$$

for some constants $a \in \mathbb{C}$, $b \in \mathbb{D} \setminus \{0\}$ *and* $c \in \mathbb{T}$.

Remark 4.2. In Schwarz's Lemma, equality in (1.3) at some point in $\mathbb{D}\setminus\{0\}$ holds if and only if equality holds at every point $z \in \mathbb{D}$. This is not true any more in Theorem 4.1. Namely, when f is the linear fractional transformation in (4.2), then equality in (4.1) occurs only for $z := 2b/(1 + |b|^2)$.

Remark 4.3. Since the origin does not play a special role, we can write (4.1) more symmetrically as follows:

$$\frac{|f(z) - f(w)|}{\operatorname{Diam} f(\mathbb{D})} \le \frac{|z - w|}{|1 - \bar{w}z| + \sqrt{(1 - |z|^2)(1 - |w|^2)}} \qquad \forall z, w \in \mathbb{D}.$$

This is done by applying Theorem 4.1 to f precomposed with a Möbius transformation, and using a well-known identity for the pseudo-hyperbolic metric.

Proof. Fix $d \in \mathbb{D}$ such that $f(d) \ne f(0)$. Set

$$g = c_1 f \circ T + c_2$$

where T is a linear fractional transformation of $\mathbb{D}$ onto $\mathbb{D}$ such that $T(x) = d$, $T(-x) = 0$, for some $x > 0$ and c_1, c_2 are constants chosen so that $g(x) = x$ and $g(-x) = -x$. By elementary algebra

$$T(z) = \frac{d}{|d|}\frac{z+x}{1+xz}$$

where $x := |d|/(1+\sqrt{1-|d|^2})$,

$$c_1 := \frac{2x}{f(d)-f(0)} \qquad \text{and} \qquad c_2 := -x\frac{f(d)+f(0)}{f(d)-f(0)}.$$

Then

$$\operatorname{Diam} g(\mathbb{D}) = |c_1| \operatorname{Diam} f(\mathbb{D}) \leq \frac{4}{|f(d)-f(0)|}\frac{|d|}{(1+\sqrt{1-|d|^2})}. \tag{4.3}$$

We now prove that $\operatorname{Diam} g(\mathbb{D}) \geq 2$ with equality if and only if $g(z) \equiv z$.

Set $h(z) := (g(z) - g(-z))/2$. Then $h(x) = x$ and $h(-x) = -x$. Note also that $h(0) = 0$ so that $h(z)/z$ is analytic in the disk and has value 1 at x and hence by the maximum principle $\sup_{\mathbb{D}} |h(z)| = \sup_{\mathbb{D}} |h(z)/z| \geq 1$ with equality only if $h(z) = z$ for all $z \in \mathbb{D}$. Since, by definition of h, $\operatorname{Diam} g(\mathbb{D}) \geq 2 \sup_{\mathbb{D}} |h|$, we see that $\operatorname{Diam} g(\mathbb{D}) \geq 2$ and then (4.3) gives (4.1) for $z = d$.

If equality holds in (4.1) at some point in $\mathbb{D} \setminus \{0\}$, then that point is an eligible d for the preceding discussion, and (4.3) shows that $\operatorname{Diam} g(\mathbb{D}) \leq 2$, while we have already shown that $\operatorname{Diam} g(\mathbb{D}) \geq 2$. Thus $\operatorname{Diam} g(\mathbb{D}) = 2$. Hence $\sup_{z \in \mathbb{D}} |h(z)| = 1$ and therefore $h(z) \equiv z$. Since h is the odd part of g, we have $g'(0) = h'(0) = 1$. Thus, by the Landau-Toeplitz Theorem 2.1 applied to g, we find that $g(z) \equiv g(0) + z$ and thus

$$f(z) = \frac{1}{c_1}T^{-1}(z) + f(T(0)).$$

Moreover, equality at $z = d$ in (4.1) says that $|f(d) - f(0)| = 2x$, hence $|c_1| = 1$. Since T is a Möbius transformation of $\mathbb{D}$, namely of the form

$$\eta\frac{z-\xi}{1-\bar{\xi}z}$$

for some constants $\xi \in \mathbb{D}$ and $\eta \in \mathbb{T}$, its inverse is also of this form. Therefore, we conclude that f can be written as in (4.2).

Finally, if f is given by (4.2), then $2b/(1+|b|^2) \in \mathbb{D} \setminus \{0\}$, and one checks that equality is attained in (4.1) when z has this value and for no other value in $\mathbb{D} \setminus \{0\}$. $\square$

5. Higher derivatives

We finish with a result, due to Kalle Poukka in 1907, which is to be compared with the usual Cauchy estimates that one gets from the maximum modulus. Interestingly, Poukka

seems to have been the first student of Ernst Lindelöf, who is often credited with having founded the Finnish school of analysis.

Theorem 5.1 (Poukka [Pou1907]). *Suppose f is analytic on $\mathbb{D}$. Then for all positive integers n we have*

$$\frac{|f^{(n)}(0)|}{n!} \leq \frac{1}{2} \operatorname{Diam} f(\mathbb{D}). \tag{5.1}$$

Moreover, equality holds in (5.1) *for some n if and only if $f(z) = f(0) + cz^n$ for some constant c of modulus* $\operatorname{Diam} f(\mathbb{D})/2$.

Proof. Write $c_k := f^{(k)}(0)/k!$, so that $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for every $z \in \mathbb{D}$. Fix $n \in \mathbb{N}$. For every $z \in \mathbb{D}$

$$h(z) := f(z) - f(ze^{i\pi/n}) = \sum_{k=1}^{\infty} c_k(1 - e^{i\pi k/n})z^k. \tag{5.2}$$

Fix $0 < r < 1$ and notice that, by absolute and uniform convergence,

$$\sum_{k=1}^{\infty} |c_k|^2 |1 - e^{i\pi k/n}|^2 r^{2k} = \int_0^{2\pi} |h(re^{i\theta})|^2 \frac{d\theta}{2\pi} \leq (\operatorname{Diam} f(\mathbb{D}))^2. \tag{5.3}$$

Therefore

$$|c_k(1 - e^{i\pi k/n})| r^k \leq \operatorname{Diam} f(\mathbb{D})$$

for every $0 < r < 1$ and every $k \in \mathbb{N}$. In particular, letting r tend to 1 and then setting $k = n$, we get $2|c_n| \leq \operatorname{Diam} f(\mathbb{D})$, which is (5.1).

If equality holds here, then letting r tend to 1 in (5.3) we get that all coefficients $c_k(1 - e^{i\pi k/n})$ in (5.2) for $k \neq n$ must be 0. Hence, $c_k = 0$ whenever k is not a multiple of n. Thus, $f(z) = g(z^n)$ for some analytic function g on $\mathbb{D}$. Moreover, $g'(0) = c_n$ and $\operatorname{Diam} g(\mathbb{D}) = \operatorname{Diam} f(\mathbb{D})$. So, by Theorem 2.1, $g(z) = cz$ for some constant c with $|c| = \operatorname{Diam} g(\mathbb{D})$, and the result follows.

∎

6. Further problems

Here we discuss a couple of problems that are related to these “diameter” questions.

The first problem arises when trying to estimate the distance of f from its linearization, $f(z) - (f(0) + f'(0)z)$, to give a “quantitative” version for the ‘equality’ case in Schwarz's Lemma (Theorem 1.1). This is done via the so-called Schur algorithm. As before, one considers the function

$$g(z) := \frac{f(z) - f(0)}{z}$$

which is analytic in $\mathbb{D}$, satisfies $g(0) = f'(0)$ and which, by assumption (1.1) and the Maximum Modulus Theorem, has $\sup_{\mathbb{D}} |g| \leq 1$. Now let $a := f'(0)$ and post-compose g with a Möbius transformation of $\mathbb{D}$ which sends a to 0 to find that

$$\frac{g(z) - a}{1 - \bar{a}g(z)} = zh(z)$$

for some analytic function h with $\sup_{\mathbb{D}} |h| \leq 1$.

Inserting the definition of g in terms of f and solving for f shows that

$$f(z) - f(0) - az = (1 - |a|^2)\frac{z^2 h(z)}{1 + \bar{a}zh(z)}$$

Thus, for every $0 < r < 1$,

$$\max_{|z|<r} |f(z) - f(0) - f'(0)z| \leq (1 - |f'(0)|^2)\frac{r^2}{1 - |f'(0)|r} \tag{6.1}$$

and 'equality' holds for at least one such r if and only if $h(z) \equiv a/|a| = f'(0)/|f'(0)|$, i.e., if and only if

$$f(z) = z\frac{a}{|a|}\frac{z + |a|}{1 + |a|z} + b$$

for constants $a \in \overline{\mathbb{D}}$, $b \in \mathbb{C}$.

In the context of this paper, when f is analytic in $\mathbb{D}$ and $\operatorname{Diam} f(\mathbb{D}) \leq 2$, by the Landau-Toeplitz Theorem 2.1 and a normal family argument we see that, for every $\epsilon > 0$ and every $0 < r < 1$, there exists $\alpha > 0$ such that: $|f'(0)| \geq 1 - \alpha$ implies

$$|f(z) - (f(0) + f'(0)z)| \leq \epsilon \qquad \forall |z| \leq r.$$

However, one could ask for an explicit bound as in (6.1).

Problem 6.1. *If f is analytic in $\mathbb{D}$ and* $\operatorname{Diam} f(\mathbb{D}) \leq 2$*, find an explicit (best?) function $\phi(r)$ for $0 \leq r < 1$ so that*

$$|f(z) - (f(0) + f'(0)z)| \leq (1 - |f'(0)|)\phi(r) \qquad \forall |z| \leq r.$$

Another problem can be formulated in view of Corollary 3.2. It is known, see the Corollary to Theorem 3 in [MW1982], that if Ω is a bounded convex domain, then the minimum

$$\Lambda(\Omega) := \min_{w \in \Omega} \rho_\Omega(w) \tag{6.2}$$

is attained at a unique point τ_Ω, which we can call the hyperbolic center of Ω. Also let us define the hyperbolic radius of Ω to be

$$R_h(\Omega) := \sup_{w \in \Omega} |w - \tau_\Omega|.$$

Now assume that $\operatorname{Diam} \Omega = 2$. Then we know, by Corollary 3.2, that $\Lambda(\Omega) \geq 1$ with equality if and only if Ω is a disk of radius 1. In particular, if $\Lambda(\Omega) = 1$, then $R_h(\Omega) = 1$.

Problem 6.2. *Given $m > 1$, find or estimate, in terms of $m - 1$,*

$$\sup_{\Omega \in \mathcal{A}_m} R_h(\Omega)$$

where $\mathcal{A}_m$ is the family of all convex domains Ω with $\operatorname{Diam}\Omega = 2$ and $\Lambda(\Omega) \leq m$.

More generally, given an analytic function f on $\mathbb{D}$ such that $\operatorname{Diam} f(\mathbb{D}) \leq 2$, define

$$M(f) := \min_{w \in \mathbb{D}} \sup_{z \in \mathbb{D}} |f(z) - f(w)|$$

and let w_f be a point where $M(f)$ is attained.

Problem 6.3. *Fix $a < 1$. Find or estimate, in terms of $1 - a$,*

$$\sup_{f \in \mathcal{B}_a} M(f)$$

where $\mathcal{B}_a$ is the family of all analytic functions f on $\mathbb{D}$ with $\operatorname{Diam} f(\mathbb{D}) \leq 2$ and

$$|f'(w_f)|(1 - |w_f|^2) \geq a.$$

References

[C1907] C. Caratheodory, *Sur quelques applications du théorème de Landau-Picard*, C. R. Acad. Sci. Paris 144 (1907), 1203-1206.

[LaT1907] E. Landau and O. Toeplitz, *Über die größte Schwankung einer analytischen Funktion in einem Kreise*, Arch. der Math. und Physik (3) 11 (1907), 302-307.

[Li1919] L. Lichtenstein, *Neuere Entwicklung der Potentialtheorie. Konforme Abbildung*, Encyklopädie der Mathematischen Wissenschaften Bd.II, 3rd Part. 1st Half, Heft 3, 181-377. B.G.Teubner (1919), Leipzig.

[MW1982] D. Minda and D. Wright, *Univalence criteria and hyperbolic metric*, Rocky Mountain J. Math. **12** (1982) no. 3, 471–479.

[PolS1972] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. 1, Springer-Verlag, 1972.

[Pou1907] K. A. Poukka, *Über die größte Schwankung einer analytischen Funktion auf einer Kreisperipherie*, Arch. der Math. und Physik (3) 12 (1907), 251-254.

[R1991] R. Remmert, *Theory of Complex Functions*, Graduate Texts in Math., vol. 122, Springer-Verlag, 1991.

Department of Mathematics, Cardwell Hall, Kansas State University, Manhattan, KS 66506, USA

E-mail address: burckel@math.ksu.edu

Department of Mathematics, Box 354350 University of Washington Seattle, WA 98195-4350 USA

E-mail address: marshall@math.washington.edu

Department of Mathematics, Cardwell Hall, Kansas State University, Manhattan, KS 66506, USA

E-mail address: pietro@math.ksu.edu